Math 246A Lecture 29 Notes

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December 5, 2018

1 Perron's Solution to the Dirichlet Problem and Regular Points

1.1 Perron's solution to the Dirichlet problem

Let Ω be a bounded domain, and let f be a bounded function on $\partial\Omega$ with $|f| \leq M$. We want to find a harmonic u on Ω such that $\lim_{z \to \zeta} u(z) = f(\zeta)$ for all $\zeta \in \partial\Omega$. This is not always possible; consider the case of a punctured disc with $f = 1$ on the boundary and 0 at the center. Instead, what we will do is describe a process for finding such a function u under given conditions.

There are several ways to do this:

1. Perron method
2. Wiener method
3. Dirichlet integrals
4. Brownian motion.

We will discuss Perron's solution.

Definition 1.1. Define $V_f = \{v : v \text{ subharmonic in } \Omega, \limsup_{z \to \zeta} v(z) \leq f(\zeta) \, \forall \zeta \in \partial\Omega\}$. This is called a **Perron family**.

Theorem 1.1. *Let $u(z) = \sup_{V_f} v(z)$. Then u is harmonic on Ω. If $f \in C(\partial\Omega)$ and if there exists u harmonic on Ω such that $\lim_{z \to \zeta} u(z) = f(\zeta)$ for all $\zeta \in \partial\Omega$, then $u(z) = \sup_{V_f} v(z)$.*

Lemma 1.1. *If $v \in V_f$, then $v \leq M$.*

Proof. Let $M' > M$, and let $E = \{z \in \Omega : v(z) \geq M'\}$. So $\operatorname{dist}(E, \partial\Omega) > 0$, and E is compact. Then $E \subseteq \tilde{\Omega}$, where $\tilde{\Omega}$ is open, and $\tilde{\Omega} \subseteq \Omega$. But $E \cap \partial\tilde{\Omega} = \varnothing$, contradicting the maximal principle. $\square$

Proof. Let $v \in V_f$, and let $B = B(z_0, R) \subseteq \overline{B(z_0, R)} \subseteq \Omega$. Then let

$$V_B = \begin{cases} v & \text{on } \Omega \setminus B \\ v & \text{on } \partial B \\ \text{solution to D.P.} & \text{on } B. \end{cases}$$

Then $v_B \leq B$ and $v \leq v_B$. Pick $z_1, z_2 \in \Omega$ with $z_1 \neq z_2$. Let $v_n \in \Omega$ be such that $v_n \in V$ satisfy $v_n(z_1) \to u_f(z)$. Let $V_n = \max(v_1, v_2, \ldots, v_n) \in V_f$. Then $\overline{B} \subseteq \Omega$, and $z_1, z_2 \in B$. Then $(V_n)_B \in V_f$ and $(V_n)_B) \uparrow u$. By the Harnack principle, u is harmonic on B

Now let $w_n \in V_f$ such that $w_n(z_2) \uparrow u_f(z_2)$. Then define the function $W_n(z) = (\max((V_n)_B, w_1, w_2, \ldots, w_n))_B$. Then $(V_n)_B \leq (U_n)_B \to \tilde{u}$ on B. Note that $i \leq \tilde{u}$ on B, and $u(z_1) - \tilde{u}(z_1)$. Then $u = \tilde{u}$ on B, so $u = \tilde{u} = u_f$ on B. Therefore, u_f is harmonic. □

1.2 Regular points

Let Ω be a bounded domain, and let $\zeta \in \partial\Omega$.

Definition 1.2. ζ is a **regular point** of $\partial\Omega$ if there exists $w(z)$ which is continuous on $\overline{\Omega}$, harmonic on Ω, $w(z) > 0$ on $\overline{\Omega} \setminus \{\zeta\}$, and $w(\zeta) = 0$.

Theorem 1.2. *Let f be bounded in $\partial\Omega$, and let $\zeta \in \partial\Omega$ be a regular point. If f is continuous at ζ, then* $\lim_{\Omega \ni z\ to \zeta} u_f(z) = f(\zeta)$.

We will prove this next time. Let's prove another result.

Theorem 1.3. *Let $\zeta \in \partial\Omega$ and $\zeta' \notin \overline{\Omega}$. If $[\zeta, \zeta'] = \{t\zeta + (1-t)\zeta' : 0 \leq t \leq 1\} \subseteq \mathbb{C} \setminus \overline{\Omega}$, then ζ is regular.*

Proof. Without loss of generality, $\zeta = -2$, and $\zeta' = 2$. We can conformally map Ω to a domain inside $\mathbb{D}$ and send the bar $[\zeta, \zeta']$ to $\partial\mathbb{D}$ via the inverse of the Joukowsky transformation, $w \mapsto w + 1/w$. □